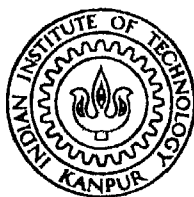# On Extremal Compactifications Of Convergence Spaces

By
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INDIAN INSTITUTE OF TECHNOLOGY KANPUR
MARCH 1975

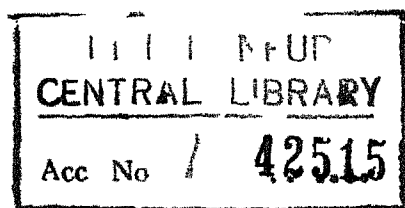# On Extremal Compactifications Of Convergence Spaces

A Thesis Submitted
In Partial Fulfilment of the Requirements
for the Degree of
DOCTOR OF PHILOSOPHY

By
CHALLA JAGAN MOHAN RAO

to the
DEPARTMENT OF MATHEMATICS
INDIAN INSTITUTE OF TECHNOLOGY KANPUR
MARCH 1975

CERTIFICATE

This is to certify that the work embodied in the thesis 'On Extremal Compactifications of Convergence Spaces' by Challa Jagan Mohan Rao has been carried out under our supervision and has not been submitted elsewhere for a degree or diploma

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ACKNOWLEDGEMENTS

CONTENTS

SYNOPSIS

Of 'On extremal compactifications of convergence spaces', a thesis
submitted in partial fulfilment of the requirements for the Ph D
degree by Challa Jagan Mohan Pao to the department of Mathematics,
Indian Institute of Technology, Kanpur

Convergence is one of the important notions in Analysis and Topology,
Beginning with the convergence of sequences, various convergences
are studied in Mathematics   The abstract theory of convergence
originated in the study of certain phenomena in Analysis and it
was recognized that there are convergences which are non-topological,
Axiomatic treatment of convergence is due to H P Fischer [4] who
laid down the foundation for an independent study into such
structures   D C Kent [7] studied a slight generalization of
Limiterung of Fischer   This generalized structure known as convergence
structure was shown subsequently by kent to be more complete one from
the point of view of the lattice of convergence structures

Our area of interest is compactification problems of convergence spaces
Quite some work has already appeared on the compactification theory
of these spaces   In this thesis our main concern is with the
extremal compactifications of convergence spaces   We study
extremal Hausdorff and regular compactifications, extremal problems
of Richardson compactification and realcompactifications   Chapter One
contains a brief summary of the known results that we need in the
present thesis

Our own work begins in Chapter Two, where we study Hausdorff compactifi-
cations   We give a characterization of the class of convergence spaces
having the smallest Hausdorff compactifications   We first construct
a one-point Hausdorff compactification that always exists, and define
the concept of local compactness for convergence spaces   We show that
a Hausdorff convergence space has smallest Hausdorff compactification
if and only if it is locally compact   In case the convergence space is
a locally compact topological space, then the one-point Hausdorff
compactification so constructed becomes topological and hence coincides
with the well known Alexandroff one-point topological compactification
We next study the problem of the existence of the largest Hausdorff
compactification. We show that if a convergence space has such a
compactification, it must be locally compact   Moreover, a necessary
and sufficient condition for the existence of the largest Hausdorff
compactification is that the convergence space has at most finitely
many non-convergent ultrafilters

In Chapter Three we study Richardson compactification [14]- a Stone-Čech
type of Hausdorff compactification of a convergence space   This has
the property that every continuous function from the convergence space
into a regular (regular we mean including Hausdorff) compact convergence
space has a unique continuous extension to this compactification   But,
in general, Richardson compactification is not necessarily regular
(and so fails to be the largest regular compactification) and also, the
regularity requirement of the range space (for extension of continuous
functions) can not be relaxed in general (and so fails to be the largest

Hausdorff compactification) Under these circumstances, it is natural
to ask when Richardson compactification is regular or the largest
Hausdorff compactification These problems are studied in this chapter
Our results supplement those of R.J. Gazik [5]

For convergence spaces, Hausdorff compactifications and regular
compactifications form two regions of study and their behaviour also
differs considerably from each other. G D Pichardson and D C Kent [15]
have characterized convergence spaces having regular compactifications,
and have further shown that such spaces also possess the largest
regular compactifications We continue this study with a characterization
of convergence spaces having smallest regular compactifications We
then define an equivalence relation on the class of all regular
compactifications of a convergence space and show that each equivalence
class has a smallest and a largest member There is a one to one
correspondence between these equivalence classes and Hausdorff topological
compactifications of the pre-topological modification of the convergence
space (this modification yielding a Tychonoff topological space)

In Chapter Five we study m-ultracompactifications of convergence spaces
m-ultracompactness for T_1 topological spaces have been defined by
J van der Slot [16], for an infinite cardinal m $\aleph_0$-ultracompactness
is equivalent to compactness and $\aleph_1$-ultracompactness is equivalent to
realcompactness for countably compact normal spaces. We extend this
to convergence spaces We construct a Hausdorff m-ultracompactification
having the extension property of continuous functions to regular m-
ultracompact space We find conditions under which it is largest

Hausdorff and largest regular ultracompactification respectively

We also find a condition under which a convergence space has smallest

m-ultracompactification

Several open problems are mentioned through out the thesis

INTRODUCTION

In this Thesis we deal with <u>convergence</u> as a primitive notion,
i e we study an abstract set in which convergence of filters is
postulated   Such a theory was originally introduced for the study
of certain non-topological phenomena in   Analysis   In 1948
Choquet [1]   in his study of relations between two non-uniformizable
topological spaces, was led in a natural way to the concept of
<u>pseudo-topology</u> on the set of closed subsets of a topological space
Thereafter Sonner [17]   gave a set of axioms for limit spaces, and
established polarity between limit spaces and topological spaces
Kowalsky [11]   studied the completion problems for such spaces
A comprehensive exposition of these concepts was given by Fischer [4],
and this is the foundation of all subsequent studies in this
direction   Kent [7] , [9] studied a slight generalization of the
Limiterung   of Fischer   This generalization known as <u>convergence</u>
<u>structure</u> was shown by Kent in his further studies to be more
complete one from the point of view of the lattice of convergence
structures   Some of the examples oi convergence structures, not
derivable from topological structures are   continuous convergence
on a collection of continuous maps from one topological space to
another (Cook and Fischer [3]), almost everywhere convergence of
measurable functions (Taylor [18]), order convergence in a lattice
(Kent [8])

Our area of interest is compactification problems of convergence
spaces Quite some work has already appeared on compactification
theory of these spaces Cochran [2] in his Doctoral Thesis
constructed a Hausdorff compactification T of a convergence space
S such that every bounded real valued continuous function from S
has a unique continuous extension to T Wyler [19] showed that
for each Hausdorff convergence space S, there exists a regular
(regular we mean including Hausdorff) compact convergence space S*,
and a continuous map j from S to S* such that every continuous map
from S into a regular compact convergence space T has a unique
continuous extension to S* But, in this case S is not necessarily
embedded in S*, and hence S* is not a compactification of S in the
usual sense Similar extensions were studied by Ramaley [12] ,
Ramaley and Wyler [13] Richardson [14] showed that every Hausdorff
convergence space S can be densely embedded in a compact Hausdorff
convergence space $(X_1, \imath)$ such that if f is a continuous function
from S into a regular compact convergence space $\imath$, then there exists
a unique continuous function h from X_1 to Y such that $h \circ \imath = f$
This compactification is called Richardson compactification In
general, this compactification is not regular, hence it fails to
be the largest regular compactification of S Moreover, the
requirement that the range space be regular (for extension of
continuous functions) can not be relaxed in general, and so this
is not necessarily the largest Hausdorff compactification These

observations lead to the investigation into the problems involving
extremal compactifications of convergence spaces, such as (i) which
convergence spaces have largest (smallest) Hausdorff compactifications,
(ii) when is Richardson compactification the largest Hausdorff
(regular) compactification, (iii) which convergence spaces have
(extremal) regular compactifications etc   In the present work, we
bring together solutions to these and other related problems

The above remarks suggest the possibility that Hausdorff
compactifications and regular compactifications constitute two
separate domains of investigation linked together by the Richardson
compactification   Our study together with those of Richardson and
Kent [15] confirms this   For this reason, we deal with these two
types of compactifications in two different chapters and a chapter
dealing Richardson compactification coming in between

In the first chapter we collect relevent notations, definitions,
and the known results required in the Thesis   The second chapter
is on Hausdorff compactifications   We begin with a characterization
of the class of convergence spaces having the smallest Hausdorff
compactifications   This is done through the construction of a
one-point Hausdorff compactification wh ch always exists, and the
concept of <u>local compactness</u> (a convergence space is locally compact
if it is open in each of its Hausdorff compactifications)   The
above mentioned characterization is   a convergence space has the

smallest Hausdorff compactification if and only if it is locally

compact In case the convergence space is a locally compact

topological space the above compactification becomes topological,

and hence coincides with the well known Alexandroff one-point

topological compactification We next obtain in this chapter a

characterization of convergence spaces having the largest Hausdorff

compactifications We show that such a convergence space is

necessarily locally compact and hence has the smallest Hausdorff

compactification In fact we prove that a convergence space has

the largest Hausdorff compactification if and only if it has at

most finitely many non-convergent ultrafilters, and hence all of

its Hausdorff compactifications are finite point compactifications

Chapter Three concerns the Richardson compactification

Gazik [5] has obtained a necessary and sufficient condition that

the Richardson compactification of a regular convergence space is

regular (and hence the largest regular compactification) A special

case of Gazik's result is the Stone-Cech compactification of a

Tychonoff topological space We continue Gazik's work and obtain

a necessary and sufficient condition that the Richardson compacti-

fication of a convergence space is its largest Hausdorff compacti-

fication Naturally this involves local compactness and some

condition on non-convergent ultrafilters

Chapter Four deals with regular compactifications Richardson
and Kent [15] have obtained a characterization of convergence
spaces having regular compactifications, and have shown that
each such convergence space has a largest regular compactification
We continue this work and obtain a characterization of those
convergence spaces which have smallest regular compactifications
Further we define an equivalence relation on the set of all
regular compactifications of a convergence space, and show that
each equivalence class has a smallest member and a largest member
And there is a one to one correspondence between these equivalence
classes and Hausdorff topological compactifications of the pre-
topological modification of the convergence space (this modifi-
cation yielding a Tychonoff topological space) This shows that
regular compactifications of a convergence space are more closely
related than Hausdorff compactification to the topological case

Chapter Five is on m-ultracompactifications m-ultracompact-
ness for T_1 topological spaces was defined by J vander Slot [16],
for an infinite cardinal m $\aleph_0$-ultracompactness is equivalent
to compactness Real compactness and $\aleph_1$-ultracompactness for
countably compact normal spaces are equivalent We extend this
definition to convergence spaces and show that it works well with
Richardson method of compactification In the first section we
find condition so that a convergence space may have smallest

m-ultracompactification and in section two we construct an m-ultracompactification in a manner similar to Richardson compactification   Then we find conditions so that this m-ultracompactification may become the largest Hausdorff m-ultracompactification and largest regular m-ultracompactification respectively   Finally we mention some open problems

CHAPTER   I

PRELIMINARIES

In this chapter we collect all notations, definitions and
results that we need in the present thesis   Most of the results
carry reference numbers indicating the place where these have been
proved   Rest of the results whose existence could not be traced
by us in the literature, are proved here   Besides these, Richardson
compactification [14] and results of Gazik [5] are described
briefly   This chapter is divided into three section, namely notations,
definitions and results

1   Notations

If S is a set and x belongs to S,   then $\underline{x}$ will denote the
ultrafilter generated by x, namely {A   S | x ε A}   Filters on
S are usually denoted by F, G etc   The set of all filters by
$F(S)$ and set of all subsets by $P(S)$ $\underline{F \wedge G}$ will denote the
filter whose base is {F ᵕ G | F ε F and G ε G}   If f is a function
from S to S', then $\underline{f(F)}$ will denote the filter on S' whose base
is {f(F) | F ε F}   If S is a subset of some set T, then filters
on T are given by ϕ, ψ etc   If every member of ϕ intersects S, that
is, trace of ϕ on S exists, then this trace is denoted by $\underline{\phi \cap S}$
The filter $\phi \cap S$ on S will generate a filter on T which is denoted

by $[\phi \cap S]_T$, when there is no possibility of doubt, the subscript

T is dropped   We would like to point out that $[\phi \cap S]_T \geq \phi$

Similarly if $S \subset T$ and F is a filter on S, then $[F]_T$ denotes the

filter on T generated by F Again we note that $[F]_T \cap S = F$

2 Definitions

A <u>convergence</u> <u>structure</u> q on a set S is a function from S

into $P(F(S))$ satisfying the following conditions

C1 x belongs to q(x) for each x in S

C2 F belongs to q(x) implies $F \wedge x$ belongs to q(x) for each x in S

C3 F belongs to q(x) and $G \geq F$ implies G belongs to q(x) for each

x in S

If F belongs to q(x), we say F <u>q-converges</u> to x   'hen the convergence

structure on S is known, we write F <u>S-converges</u> to x, or simply F

<u>converges</u> to x when there is no possibility of doubt   A subset A

of S is said to be <u>open</u> if each filter F on S which converges to a

point x in A, contains A   A is said to be <u>closed</u> if complement of

A in S is open   The set of all closed subsets of S is denoted by

$\underline{C_S}$ For a filter F on S $\underline{F \cap C_S}$ denotes the set of all closed subsets

of S in F A subset B of S is said to be the <u>closure</u> of a subset

A of S if B is the set of all x in S such that there exists a

filter F on S containing A converging to x   The closure of A in S

is denoted by $\underline{C\ell_S A}$ For a given filter F on S, $\underline{C\ell_S F}$ is the filter

which is generated by $\{C\ell_S F \mid F \in F\}$ For x in S we denote by $\underline{V_S(x)}$

the intersection filter of all filters that converge to x   A point

x in S is said to be <u>adherence point</u> of a filter F on S   if there

exists an ultrafilter $G \geqslant F$ such that G converges to x   A convergence

structure is said   to be <u>Hausdorff</u> if each filter on S converges at

most to one point   A Hausdorff convergence space is said to be

<u>regular</u> if F converges to x implies $Cl_S \, F$ converges to x   If F is an

ultrafilter on   S, then F is said   to be <u>m-ultrafilter</u> for an infinite

cardinal m, if $F \cap C_S$ has m-intersection property (i e   each subcolle-

ction of closed subsets of F of cardinal $< m$ has non-empty intersection)

S is said to be <u>m-ultracompact</u> if each m-ultrafilter on S converges

$\aleph_0$-ultracompact space is said to be <u>compact</u>   Therefore, S is compact

if and only if every ultrafilter on S converges   A function f from

a convergence space S to another S' is said to be <u>continuous</u> if a filter

F on S is S-convergent to x in S,   then $f(F)$ is S'-convergent to $f(x)$

A one-to-one bicontinuous function from S to S' is said   to be

<u>isomorphism</u> and S and f(S) are said to be <u>isomorphic</u>   A subset S of

a convergence space T is said to be <u>dense</u> in T if for each x in T, there

exists a filter on T converging to x and containing S   If S and S'

denote two convergence spaces on the same set, we say <u>S is finer than S'</u>

(or <u>S' is coarser than S</u>) if for each filter F on S, F S-converges to

x implies F S'-converges to x, and it is denoted by <u>S $\geq$ S'</u>   Under this

ordering it is easily seen that the set of all   convergence structures

of a set is a complete lattice, where if $C(S)$ is a set of such structures,

then <u>inf $C(S)$</u> and <u>sup $C(S)$</u> are given as follows

a filter F-inf $C(S)$ - converges to x if there exists a convergence

structure q in $C(S)$ such that F q-converges to x,

F-sup $C(S)$ - converges to x if F q-converges to x for each q in $C(S)$

If S is a Hausdorff convergence space and T a compact Hausdorff

convergence space and there exists a bicontinuous one to one function

from S into T such that $f(S)$ is dense in T, then T is said to be a

Hausdorff compactification of S, and is denoted by (T,f) If (T,f)

and (T',g) are two Hausdorff compactifications of S, then we say

(T,f) is finer than (T',g) and denote $(T,f) \geq (T',g)$ if there exists

a continuous function h from T onto T' such that h o f = g We say

(T,f) and (T',g) are equivalent compactifications of S if

$(T,f) \geq (T',g)$ and $(T',g) \geq (T,f)$ Hausdorff compactification (T,f)

of S is said to be regular if T is a regular convergence space If S

is a Hausdorff convergence space and T an m-ultracompact Hausdorff

space, and f is a bicontinuous one to one function from S to T such

that $f(S)$ is dense in T, we say (T,f) is an m-ultracompactification

of S If (T,f) and (T',g) are two m- ultracompactifications of S,

then we write $(T,f) \geq (T',g)$ if there exists a continuous function

h from T into T' such that h o f = g, and say (T,f) is finer than

(T',g) If S is a convergence space such that $V_S(x)$ converges to x

for each x, we say S is a pretopological space The finest pre-

topological space on the set S coarser than S is denoted by πS This

is defined as F , a filter on S will πS-converge to x in S if and

only if $F \geq V_S(x)$ This is called pretopological modification of S

Similarly the finest topology on the set S coarser than S is called

the topological modification of S and is denoted by λS λS is

defined as   a filter F on S will λS-converge to x in S if and only

if $F \geq B_S(x)$, where $B_S(x)$ is the filter generated by the sets U

in $V_S(x)$, which have the property that y belongs to U implies U

belongs to $V_S(y)$ If S is a subset of a convergence space T then

subspace convergence structure on S is defined as follows   a filter

F on S will S-converge to x in S if and only if $[F]_T$ T-converges to x

3   Results

We begin with two simple results of a technical nature   We

will make use of these often without explicit mention

Theorem 1 1   If F_1 and F_2 are two filters on a set S and G is

an ultrafilter on S such that $G \geq F_1 \wedge F_2$, then $G \geq F_1$ or $G \geq F_2$

Proof   Let F_1 be in F_1 and F_2 in F_2, such that both F_1 and F_2 does

not belong to G   Now $F_1 \cup F_2$ belongs to $F_1 \wedge F_2$ and therefore to G

But G is an ultrafilter, hence either F_1 or F_2 belongs to G Hence

we get a contradiction

Theorem 1 2   A set S is finite if and only if every ultrafilter

on S is a point ultrafilter

Proof   Let S be a finite set and F be an ultrafilter, then S belongs

to F, hence some {x} for x in S belongs to S,   that is, F is a point

ultrafilter   On the other hand if S is infinite   then S-complements

of finite sets in S will constitute a filter base, let F be an

ultrafilter finer than this, then F is not a point ultrafilter

__Theorem__ 1 3 [Fischer [4], Theorem 10, p 278] If S is a pre-topological compact convergence space and S* is a Hausdorff convergence space such that S* $\leq$ S, then S* = S

__Theorem__ 1 4 [Fischer [4], Theorem 13, p 279] If S is a convergence space and x belongs to $C\ell_S$ A for A $\subset$ S, then U $\cap$ A $\neq \emptyset$, for every U in $V_S(x)$

__Theorem__ 1 5 [Kent [6], Lemma 1, p 3] If S is a convergence space, then A $\subset$ S is S-closed if and only if A is λS-closed if and only if $C\ell_S$ A = A

__Theorem__ 1 6 [Fischer [4], Theorem 15, p 279] If S is a Hausdorff convergence space, then for each x in S, {x} is closed

__Theorem__ 1 7 [Kent [6], Theorem 2, p 5] If S is a compact Hausdorff space and A is a closed subset of S, then A is compact

__Theorem__ 1 8 [Kent [6], Theorem 2, p 5] If S is a Hausdorff convergence space and A is a compact subset of S, then A is closed in S

__Theorem__ 1 9 [Kent [6], Theorem 1, p 16] Let x be in A $\subset$ S, then $V_S(x) \cap A = V_A(x)$

__Theorem__ 1 10 [Kent and Richardson [10], Theorem 1 1 p 488] A convergence space S is minimal Hausdorff if and only if S is a compact Hausdorff convergence space such that whenever every ultrafilter finer than F converges to some x in S, then so does F

Theorem 1 11 [Fischer [4], Theorem 1, p 283] Composition of two

continuous functions is continuous

Theorem 1 12 [Fischer [4], Theorem 2, p 283] Restriction of a

continuous function to a subspace is continuous

Theorem 1 13 [Kent [6], Theorem 4, p 9] Continuous image of a

compact space is compact

Theorem 1 14 [Fischer [4], Theorem 5, p 284] If f is a continuous

function from a convergence space S to another S*, then $f(Cl_S A) \subset Cl_{S*} f(A)$,

for each A S

Theorem 1 15 [Kent [6], Theorem 2, p 8] If f is a continuous

function from S to S*, then A $\frown$ S* is S*-closed implies $f^{-1}(A)$ is S-

closed

Theorem 1 16 [Kent [9], Corollary to Theorem 1, p 199] If f is a

continuous function from S to S*, then so is f from πS to πS*

Theorem 1 17 [Kent [7], Theorem 4, p 130] If S is a convergence

space, then S-closure operator is idempotent if and only if πS is a

topology

Theorem 1 18 [Kent and Richardson [10], Theorem 1 9, p 490] If

f is a continuous bijection from a minimal Hausdorff space into a

Hausdorff space S*, then f is an isomorphism

Theorem 1 19 [Fischer [4], Theorem 6, p 284] If f and g are two

continuous functions from S to S* such that f and g agree on a dense

subset A of S and S* is Hausdorff, then f = g

From this theorem we derive the following two results which we will use often

__Theorem__ 1 20   If S is a dense subspace of a converges space T   Then a continuous function from S into a Hausdorff convergence space S* has at the most one extension to T,   that is, whenever such an extension exists, it is unique

__Theorem__ 1 21   Let (T_1, f_1) and (T_2, f_2) be two Hausdorff compactifications (or m-ultracompactifications) of a convergence space S such that each is finer than the other, then the two are equivalent, that is, there exists an isomorphism between T_1 and T_2 such that $h \circ f_1 = f_2$

__Theorem__ 1 22   Let S be a convergence space and T_1, T_2 be Hausdorff convergence spaces and f_1 and f_2 be isomorphisms from S to T_1 and T_2 such that $f_1(S)$ and $f_2(S)$ are dense in T_1 and T_2 respectively   Let h be a continuous function from T_1 to T_2 such that $h \circ f_1 = f_2$ Then $h(f_1(S)) = f_2(S)$ and $h(T_1 - f(S)) \subseteq T_1 - f_2(S)$, and when h is an onto function then $h(T_1 - f(S)) = T_2 - f_2(S)$

__Proof__   The first property follows from the fact $h \circ f_1 = f_2$ To prove the second we need only show that $h(T_1 - f_1(S)) \cap h(f_1(S)) = \phi$ Let $S_1 = h^{-1} \circ h(f_1(S))$, it is sufficient to show that $S_1 = f_1(S)$ Now $f_1(S) \subset S_1 \subset T_1$, S_1 is Hausdorff and $f_1(S)$ is dense in S_1 Let g be the inverse of the map h restricted to $f_1(S)$ and g_1 be $g \circ (h| S_1)$, then g_1 restricted to $f_1(S)$ is identity and hence g_1 is identity from S_1 to S_1 Since $g(S_1) \subset f_1(S)$ we have $S_1 = f_1(S)$

$$T_1 \xrightarrow{\quad h \quad} T_2$$

$$f_1 \qquad\qquad f_2$$

$$S$$

This result we will use in the context of compactifications (or m-ultracompactifications in general)

Theorem 1 23 [Richardson and Kent [15], Proposition 1, p 572] The closure operator of a compact regular convergence space is idempotent

<u>Theorem</u> 1 24 [Richardson and Kent [15], Proposition 3, p 572] If A is subspace of a compact regular convergence space, then πA is Hausdorff and topological

Now we describe Richardson compactification [14], of a Hausdorff convergence space S   Let X_1 be the set of all x for x in S and all nonconvergent ultrafilters on S   Let ı be the mapping from S to X_1 defined as $ı(x) = x$ Let F be a filter on S, then $\hat{F}$ is the filter on X_1 whose base is $\{\hat{F} \mid F \in F\}$, where $\hat{F} = \{H \in X_1 \mid \Gamma \in H\}$ A convergence structure is defined on the set X_1 (and the resulting space is denoted by X_1 as follows

let ϕ be a filter on X_1, then

ϕ X_1-converges to x if and only if $\phi \geq \hat{F}$ for some filter F on S which S-converges to x

ϕ X_1-converges to v if and only if $\phi \geq \hat{v}$, where v is a nonconvergent ultrafilter on S

The space $(X_1, ı)$ is the Richardson compactification

In case X_1 consists of all x for x in S and all nonconvergent

m-ultrafilters on S, then with a convergence structure defined

same as above is an m-ultracompact space   (Chapter Five of this thesis)

We will now give proof of the following two results which have been proved by Gazik [5]   The proofs are valid for both Richardson compactification and m-ultracompactification described above, though Gazik dealt only the compactification case

Let us denote by ρS the Richardson compactification (or m-ultracompactification described above) of S   Let ϕ be a filter on ρS If A belongs to ϕ and f is any function from A to P(S) for which $f(D) \in D$ for each $D \in A$, we define $A(f) = \cup\{f(D) \mid D \in A\}$ Since ϕ is a filter on ρS, $A(f)$ for $A \in \phi$, constitute a filter base on S, let us denote it by $K(\phi)$ the filter generated by this base

Theorem 1 25   [Gazik [5], Lemma 1, p 2]   Let ϕ be a filter on ρS, $\alpha \in \rho S$ and ϕ ρS-converges to α Then $K(\phi) = \alpha$, if α is non-convergent and $K(\phi)$ S-converges to x if $\alpha = x$

Proof Let α be nonconvergent If $A \in \alpha$, by definition ρS,   there exists an $H \in \phi$ such that $H \subset \hat{A}$ If f is defined on H as $f(h) = A$ for each h in H, then $H(f) = \cup\{f(h) \mid h \in H\} = A \in K(\phi)$ Hence $K(\phi) \geq \alpha$ But α is ultrafilter, hence $\alpha = K(\phi)$ If $\alpha = x$, then $\phi \geq \hat{F}$ for some filter F on S which S-converges to x Hence similar to above $K(\phi) \geq \alpha$

Theorem 1 26 [Gazik [5], Lemma 2, p 2]   If $A \subset S$, then
$$c\ell_{\rho S}(\hat{A}) \subset (c\ell_S(A))^{\wedge}$$

<u>Proof</u>   Let $\alpha \, \epsilon \, Cl_{\rho S}(\hat{A})$ and α is nonconvergent   Then there exists

a filter ϕ on ρS such that $\hat{A} \, \epsilon \, \phi$ and ϕ ρS-converges to α Hence

by Theorem 1 25, $K(\phi) = \alpha$, so $\hat{A}(f) \, \epsilon \, K(\phi) = \alpha$, where f is defined

on $\hat{A}$ by $f(h) = A$ for $h \, \epsilon \, \hat{A}$ Thus $A = \hat{A}(f) \, \epsilon \, \alpha$ It follows that

$\alpha \, \epsilon \, \hat{A}$, so $\alpha \, \epsilon \, (Cl_S(A))^{\wedge}$ Let $\alpha = x \, \epsilon \, Cl_{\rho S}(\hat{A})$ Then there is a

filter ϕ on ρS such that ϕ ρS-converges to x and $\hat{A} \, \epsilon \, \phi$ By

Theorem 1 25, $K(\phi)$ S-converges to x With f defined on $\hat{A}$ by

$f(h) = A$ for $h \, \epsilon \, \hat{A}$, we have $A = \hat{A}(f) \, \epsilon \, K(\phi)$ Hence $x \, \epsilon \, Cl_S(A)$,

therefore, $x \, \epsilon \, (Cl_S(A))^{\wedge}$

CHAPTER II

On HAUSDORFF COMPACTIFICATIONS

Introduction

Since every Hausdorff convergence space has a Hausdorff compactification, characterization of the class of Hausdorff convergence spaces having smallest Hausdorff compactifications, or likewise having largest Hausdorff compactifications, is of some interest   In this chapter we investigate such problems   Firstly, in section one we obtain a one point Hausdorff compactification of a convergence space   This compactification always exists without needing any additional conditions on the underlying convergence space   Using this one point compactification, we obtain a characterization of the class of convergence spaces which possess the smallest Hausdorff compactifications   This characterization is in terms of local compactness, a notion that has been defined in this section   This definition of local compactness which has been defined in analogy with topological case, is an 'external' one, in the sense that we demand the space to be an open subspace of each of its Hausdorff compactifications   We do not know any 'internal' characterization, it has been proposed here as an open problem   We also show that the one point compactification constructed in this section is 'proper', in the sense that if we take the underlying convergence space to be a locally compact Hausdorff topological space, then this

compactification becomes topological, and hence coincides with the Alexandroff one point compactification In section two we obtain a characterization of the class of Hausdorff convergence spaces which have the largest Hausdorff compactifications For this we use the Richardson compactification [14] , construction of which has been outlined in Chapter One Firstly, we show that any convergence space having the largest Hausdorff compactification is locally compact in the sense defined in this chapter And hence, each such convergence space also possesses the smallest Hausdorff compactification In fact we show that a convergence space has the largest Hausdorff compactification if and only if it has atmost finitely many non-convergent ultrafilters, as a consequence all of its Hausdorff compactifications are finite point compactifications

1   *Smallest Hausdorff Compactification*

In this section our aim is to obtain a characterization of the class of Hausdorff convergence spaces which possess the smallest Hausdorff compactifications

The one point Hausdorff compactification

Let S be a Hausdorff non-compact convergence space   Let $T = S \cup \{z\}$, where z does   not belong to S We make T a convergence space by

defining a convergence structure as follows

let ϕ be a filter on S, then

ϕ T-converges to z if and only if $\phi \geq \psi \wedge z$, where ψ is any

filter on T such that $\psi \cap S$ has no S-adherence points

ϕ T-converges to x in S if and only if S belongs to ϕ and ϕ 'S

S-converges to x

Let i be the inclusion map from S into T We will show that

(T, i) is a Hausdorff compactification of S The fact that T

is a convergence space is simple to verify Let ϕ be an ultra-

filter on T which is T-converging to some x in S and also T-

converging to z Then, $\phi \geq \psi \wedge z$, where ψ is a filter on T

such that ψ ' S has no S-adherence points Since ϕ T-converges

to x in S, $\phi \neq z$, therefore $\phi \geq \psi$ But this is a contradiction

Hence, T is a Hausdorff convergence space In order to see that

T is compact, let ϕ be an ultrafilter on T If $\phi = z$, we are

through If $\phi \neq z$, then S belongs to ϕ Now, $\phi \cap S$ either S-

converges to some x in S, or is a non-convergent ultrafilter on

S In the first case ϕ T-converges to x and in the second case

ϕ T-converges to z i is obviously a bicontinuous one to one

function from S into T Since S is a non-compact convergence

space, it has a non-convergent ultrafilter F If we denote by

ψ, the filter generated by $i(F)$ on T, then by definition, ψ T-

converges to z, and hence $i(S)$ is dense in T

Local compactness

We give a definition of local compactness for a Hausdorff
convergence space which is analogous to the topological case
We would like to recall that a Tychonoff topological space is
locally compact if and only if it is open, as a subspace, in
each of its Hausdorff topological compactifications

Definition 2 1   A Hausdorff convergence space is locally compact
if and only if it is open, as subspace, in each of its Hausdorff
compactifications

Open problem   Find an internal characterization of local compact-
ness for convergence spaces

Characterization

Now we are in a position to give a characterization of the class
of Hausdorff convergence spaces having the smallest Hausdorff
compactifications

Theorem 2 2   A Hausdorff non-compact convergence space has the
smallest Hausdorff compactification if and only if it is locally
compact

<u>Proof</u>   Let S be a Hausdorff non-compact locally compact convergence

space, and $(T, ı)$ be the one point Hausdorff compactification of

S constructed above   We will show that in this case $(T, ı)$ is

the smallest Hausdorff compactification of S   Let (T',f) be an

arbitrary Hausdorff compactification of S   Let us define a

function h from T' onto T as follows

$$h(y) = ı \circ f^{-1}(y) \quad , \quad \text{for y in } f(S)$$

$$= z \qquad\qquad , \quad \text{for y in } T' - f(S)$$

Therefore h is a function such that the following diagram commutes

$$
\begin{array}{ccc}
T' & \xrightarrow{\ h\ } & T \\
 & \nwarrow \quad \nearrow & \\
 & f \quad\ \ ı & \\
 & S &
\end{array}
$$

We will show that h is a continuous function from T' onto T   The

fact that the function is well defined is obvious   Let ϕ be a

filter on T' which is T'-converging to y in $f(S)$ Now, S being

locally compact, $f(S)$ belongs to ϕ Hence, $f^{-1}(\phi \restriction f(S))$ S-

converges to $f^{-1}(y)$ Therefore, $ı \circ f^{-1}(\phi \restriction f(S))$ T-converges to

$ı \circ f^{-1}(y) = h(y)$ Since $h(\phi) = ı \circ f^{-1}(\phi \restriction f(S))$, we are through

Next, let ϕ T'-converge to y in $T'-f(S)$ We have the following

two possibilities   (a) the trace of ϕ on $f(S)$ does not exist,

then $T'-ı(S)$ belongs to ϕ, hence $h(\phi) = z$ which T-converges to

$z = h(y)$, (b) the trace of ϕ on $f(S)$ exists, then $\phi \cap f(S)$ has

no $f(S)$-adherence points because ϕ T'-converges to y in $T'-f(S)$

and T' is Hausdorff Therefore, $f^{-1}(\phi \| f(S))$ has no S-adherence

points, and in turn, $i \circ f^{-1}(\phi \| f(S))$ has no $i(S)$-adherence points

But $i \circ f^{-1}(\phi \cap f(S)) = h(\phi \cap f(S)) = h(\phi) \cap i(S)$, hence $h(\phi)$ T-

converges to $z = h(y)$ Since $h \circ f = i$, and the uniqueness of h

follows from Theorem 1 20, we have $(T',i) \geq (T,i)$

Next, let S be a non-compact convergence space having the smallest

Hausdorff compactification Let (T_1,g_1) be the smallest Hausdorff

compactification of S Since, the one point Hausdorff compact-

ification (T,i) always exists, we have $(T,i) \geq (T_1,g_1)$ Now, by

Theorem 1 22, we conclude that (T_1,g_1) is a one point compactifi-

cation of S Hence, S is open in its smallest Hausdorff compact-

ification We will now show that S is open in each of its Hausdorff

compactifications This we do as follows let (T_2,g_2) be an

arbitrary Hausdorff compactification of S Since $(T_2,g_2) \geq (T_1,g_1)$,

there exists a continuous function h from T_2 onto T_1 such that

$h \circ g_2 = g_1$ Now, by Theorem 1 22, $h^{-1}(T_1-g_1(S)) = T_2-g_2(S)$ And

$T_1-g_1(S)$ is T_1-closed being a singlton by Theorem 1 6, hence by

Theorem 1 15, $T_2-g_2(S)$ is T_2-closed, and thus $g_2(S)$ is open

implying thereby that S is locally compact This completes the

proof of the theorem

Remarks on the one point compactification

We note the following two facts, which we obtain by observing the

proof of the above theorem

(a)   The one point Hausdorff compactification $(T,1)$ of a convergence space S is coarser than any Hausdorff compactification of S in which S is open   This follows from the proof of 'if' part of the above theorem   This implies, in particular, that $(T,1)$ is coarser than any finite point compactification of S   On the other hand if some Hausdorff compactification (X,f) of S is such that $(X,f) \geq (T,1)$, then using Theorem 1 22, one can show that S is open in X

Theorem 2 3   A Hausdorff compactification (X,f) of S is finer than the one point Hausdorff compactification $(T,1)$ of S if and only if S is open in X

(b)   If some Hausdorff compactification of S is coarser than the one point Hausdorff compactification $(T,1)$ of S, then it follows from the proof of the 'only if' part of the Theorem 2 2, that this compactification must be a one point compactification   But in such a case, as it has been pointed out in (a) above, it must be finer than $(T,1)$, and hence by Theorem 1 21, it is equivalent to $(T,1)$

Theorem 2 4   If some Hausdorff convergence space has the smallest Hausdorff compactification, then it must be $(T,1)$

Topological case

Let S be a locally compact (topological sense) Hausdorff topological space Let us denote by $(T',\jmath)$ the Alexandroff one point Hausdorff topological compactification of S Let $(T,\imath)$ be the one point Hausdorff compactification (convergence sense) of S as constructed earlier Then, by remark (a) above $(T',\jmath) \geq (T,\imath)$ Now, T' is a compact Hausdorff topological space, hence by Theorem 1 10, T' is a minimal Hausdorff space in the convergence sense Let h be the continuous function from T' onto T such that $h \circ \jmath = \imath$ Now, both T' and T being one point compactifications of S, by Theorem 1 22, h is a continuous bijection from T' onto T Therefore, by Theorem 1 18, h is an isomorphism and hence $(T',\jmath) = (T,\imath)$

Theorem 2 5   If S is a locally compact Hausdorff topological space, then $(T,\imath)$ is its Alexandroff one point Hausdorff topological compactification

Now combining Theorem 2 4 and Theorem 2 5, we have the following result

Theorem 2 6   If a locally compact Hausdorff topological space has a smallest Hausdorff compactification in the convergence sense, then this compactification is topological, and in fact is the Alexandroff one point compactification

2   Largest Hausdorff Compactification

In this section we obtain a characterization of the class of Hausdorff
convergence spaces which possess the largest Hausdorff compactifications

Local compactness

Firstly, we prove that any Hausdorff convergence space having the
largest Hausdorff compactification is necessarily locally compact,
and hence also possesses the smallest Hausdorff compactification

Theorem 2 7   If a Hausdorff convergence space has the largest
Hausdorff compactification, then it is locally compact

Proof   Let S be a Hausdorff convergence space having the largest
Hausdorff compactification (X,f) Let (T,i) denote the one point
Hausdorff compactification of S constructed in section one   Then
by Theorem 2 3, we know that $f(S)$ is open in X   Now, we will show
that S is open in each of its Hausdorff compactifications   Let
(Y,g) be an arbitrary Hausdorff compactification of S   Then
$(X,f) \geq (Y,g)$ From Theorem 1 22, we know that if h is the continuous
function from X onto Y such that $h \circ f = g$,   then $h(X-f(S)) = Y-g(S)$
Now, $f(S)$ being open, $X-f(S)$ is closed, and by Theorem 1 7, it is
compact   Hence $Y-g(S)$ being a continuous image of a compact set is

compact by Theorem 1 13   Since Y is Hausdorff, by Theorem 1 8,
Y-g(S) is closed, and thus g(S) is open in Y   This proves S is
locally compact

Characterization

Now we will obtain a characterization of the class of Hausdorff
convergence spaces having the largest Hausdorff compactifications

Theorem 2 8   A Hausdorff convergence space S has the largest
Hausdorff compactification if and only if S has at most finitely
many non-convergent ultrafilters

Proof   We first prove the necessary part   The proof follows in
the following steps   (i) Let (X,i) be the largest Hausdorff
compactification of S   Then, from Theorem 2 7, we know that S
is locally compact   (ii) Let (X_1,i) be the Richardson compactifi-
cation of S   Then $(X,f) \geq (X_1,i)$ Let h be the continuous function
from X onto X_1, such that $h \circ f = i$ Let us define a map g from
X_1 to X as follows

$g(x) = f(x)$,   for x in S

$g(v) = \lim f(v)$,   for a non-convergent ultrafilter v on S

Here lim stands for the unique limit to which the ultrafilter $f(v)$
on the compact set X will converge
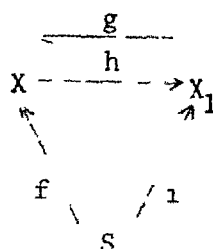
This function g is well defined   We will show that g is onto X

and is one to one   Let y belongs to X, since f(S) is dense in X,

there exists an ultrafilter ϕ on X containing f(S) which X-

converges to y   Now $f^{-1}(\phi \cap f(S))$ is a non-convergent ultrafilter

on S, and hence, is an element of X_1-1(S)   Therefore,

$g(f^{-1}(\phi (f(S)))) = \lim f(f^{-1}(\phi \cap f(S))) = \lim \phi = y$ Hence, g

is onto X   Now, $h(g(x)) = h(f(x)) = 1(x) = x$, and

$h(g(v)) = h(\lim f(v)) = \lim (h(f(v)) = \lim 1(v) = v$ Hence g

is one to one   Now, this function g enables us to identify X

and X_1 as sets   Hence, we can consider (X,f) to be consisting

of the set X_1 and f to be defined as $f(x) = x$ from S into X

(iii) Let us define a new convergence structure on X (and denote

the new space by X') as follows

let ϕ be a filter on X,   then

ϕ X'-converges to x if and only if ϕ X-converges to x

ϕ X'-converges to v if and only if $\phi \geq v \wedge \psi$, where ψ is an

ultrafilter on X   which X-converges to v

We obviously see that (X',f) is a Hausdorff compactification of S

and that $X' \geq X$, implying thereby that $(X',f) \geq (X,f)$ But by

assumption $(X,f) \geq (X',f)$ Hence, by Theorem 1 21, $(X,f) = (X',f)$

and thereby $X = X'$ From this we see that a filter ϕ on X will X-converge to v if and only if $\phi \geq v \wedge \psi$, where ψ is an ultrafilter on X which X-converges to v (iv) Let v be a non-convergent ultrafilter on S, then $f(v)$ is an ultrafilter on X, and hence must X-converge But $f(v)$ cannot converge to a point in $f(S)$, let it converge to some v' in $X - f(S)$ Then $h \circ f(v)$ will X_1-converge to $h(v')$ That is, $i(v)$ X_1-converges to $h(v')$ But $i(v)$ X_1-converges to v and X_1 is a Hausdorff convergence space, and therefore $h(v') = v$ Hence, $g \circ h(v') = g(v) = v$ But $g \circ h =$ Identity on X , $v' = v$ Now, $f(v)$ X-converging to v, implies from (iii), that $f(v) \geq v \wedge \psi$, where ψ is an ultrafilter which X-converges to v Since $f(v) \neq v$, we have $f(v) = \psi$ Thus, we see that a filter ϕ on X will X-converge to v if and only if $\phi \geq v \wedge f(v)$ (v) Let ϕ be an ultrafilter on X containing $X - f(S)$ ϕ must X-converge, but cannot converge to a point in $f(S)$ because S is locally compact as mentioned in step (i) above, and hence $f(S)$ is X-open and ϕ contains $X - f(S)$ Let ϕ X-converge to some v in $X - f(S)$ Then as shown in step (iv) above, $\phi \geq v \wedge f(v)$ Since $\phi \neq f(v)$, we conclude that $\phi = v$ That is each ultrafilter on $X - f(S)$ is a point filter and hence by Theorem 1 2, $X - f(S)$ is finite, that is, S has at most finitely many non-convergent ultrafilters

Next, to prove the sufficiency part, let us assume that S has only finitely many non-convergent ultrafilters Let us denote by Y, the set of all x for x in S and all non-convergent ultrafilters on S

Let k be a function from S to Y defined as $k(x) = x$ Let us define

a convergence structure on Y(and denote the resulting space by Y)

as follows

let ϕ be a filter on Y, then

ϕ Y-converges to x if and only if k(S) belongs to ϕ and

$k^{-1}(\phi \lceil_i k(S))$ S-converges to x in S

ϕ Y-converges to v if and only if $\phi \geq v \wedge k(v)$, for non-convergent

ultrafilters v on S

It is easy to verify that Y is a convergence space   To see that

it is Hausdorff, let ϕ be an ultrafilter on Y which is simultane-

ously Y-converging to (a) x_1 and x_2 Then $k^{-1}(\phi_{,i} k(S))$ S-

converging to x_1 and x_2, but S is Hausdorff, hence $x_1 = x_2$,

(b) x and v, where v is a non-convergent ultrafilter on S   Then

k(S) belongs to ϕ and $k^{-1}(\phi \cap k(S))$ S-converges to x, and

$\phi \geq v \wedge k(v)$ This implies $\phi \neq v$, hence $\phi = k(v)$ But then,

$k^{-1}(\phi \cap k(S)) = v$, which is a non-convergent ultrafilter on S,

hence a contradiction, (c) v_1 and v_2, where v_1 and v_2 are two

non-convergent ultrafilters on S   Then $\phi \geq v_1 \wedge k(v_1)$ and

$\phi \geq v_2 \wedge k(v_2)$ If $v_1 \neq v_2$, then $\phi \neq v_1$ and $\phi \neq v_2$, this implies

that $\phi = k(v_1)$ and $\phi = k(v_2)$ But then, by definition of k, we

see $v_1 = v_2$, a contradiction   To see that Y is compact, let ϕ be

an ultrafilter on Y   If Y-k(S) belongs to ϕ, then since Y-k(S) is

finite, by Theorem 1 2, ϕ is a point filter for some point in Y-k(S),

and hence convergent   If k(S) belongs to ϕ , then (a) if

$k^{-1}(\phi \cap k(S))$ S-converges to some x in S, then by definition ϕ Y-converges to x, (b) if $k^{-1}(\phi \cap k(S))$ is non-convergent on S, then this being an ultrafilter, ϕ will Y-converge to $k^{-1}(\phi \cap k(S))$ by definition Obviously the function k is a bicontinuous one to one function from S into Y k(S) is dense in Y, because for v in Y-k(S), k(v) contains k(S) and Y-converges to v by definition

Therefore, we find that (Y,k) is a Hausdorff compactification of S Now, we will show that (Y,k) is the largest Hausdorff compactification of S To show this we will prove that if Z is a compact Hausdorff convergence space and s is a continuous function from S into Z, then this function s can be extended to a unique continuous function from Y onto the closure of s(S) in Z For this, let us define a function t from Y to Z as follows

$t(x) = s(x)$, for x in S

$t(v) = \lim s(v)$, for v in Y-k(S)

Now, we can verify that t is a well defined function onto the closure of s(S) in Z such that $t \circ k = s$ To see that t is continuous, let ϕ be a filter on Y which Y-converges to x Then, by definition k(S) belongs to ϕ and $k^{-1}(\phi \cap k(S))$ S-converges to x Hence, $s \circ k^{-1}(\phi \cap k(S))$ Z-converges to s(x), and hence, $t(\phi)$ Z-converges to t(x) Let ϕ Y-converges to v for some non-convergent

ultrafilter v on S, then $\phi \geq v \wedge k(v)$, and thererore, $t(\phi) \geq t(v \wedge k(v))$

$= t(v) \wedge t \circ k(v) = t(v) \wedge s(v)$, which Z-converges to $t(v)$, and hence,

$t(\phi)$ Z-converges to $t(v)$ Now, uniqueness of the function t follows

from Theorem 1 20 This completes the proof of the theorem

CHAPTER III

ON THE RICHARDSON COMPACTIFICATION

Introduction

Richardson constructed a compactification of a convergence

space in [14]   This was out lined in Chapter One   This is a

Hausdorff compactification, showing also that every Hausdorff

convergence space has a Hausdorff compactification   This compacti-

fication has the following property   let S be a Hausdorff convergence

space and (X_1, ι) be its Richardson compactification   Let f be

a continuous function from S into a regular compact convergence

space Y   Then, there exists a continuous function h from X_1 onto

the closure of f(S) in Y such that h∘ι = f

$$X_1 \xrightarrow{\ h\ } Y$$
$$\iota \diagdown \quad \diagup f$$
$$S$$

Now, we note the following two things (a) (X_1, ι) is, in general,

not regular, and therefore, fails to be the largest regular

compactification of S, (b) the regularity requirement on the

convergence space Y cannot be relaxed, and therefore, (X_1, ι)

fails to be the largest Hausdorff compactification of S

These observations make us study Hausdorff as well as regular compactifications of convergence spaces with Richardson compactification appearing in both   In the domain of regular compactifications, the problem is under what additional conditions, will $(X_1, 1)$ become regular, and thereby, the largest regular compactification   This problem has been solved by Gazik in [5] , where he has obtained a necessary and sufficient condition for $(X_1, 1)$ to be regular   This condition is in terms of some requirement on non-convergent ultrafilters on the underlying convergence space   He has further shown that his condition is also both necessary and sufficient for the Richardson compactification of a Tychonoff topological space may become topological, and thereby, coincide with the Stone-Čech compactification   In section one of this chapter we mention Gazik's results without proof   Next, in the domain of Hausdorff compactifications, the problem is, to characterize the class of Hausdorff convergence spaces for which the Richardson compactification becomes the largest Hausdorff compactification In section two we obtain such a characterization   This characterization is in terms of local compactness as defined in the previous chapter and some condition on non-convergent ultrafilters on the underlying convergence space

1 *As The Largest Regular Compactification*

Let S be a regular convergence space   Let us denote by

R(S) its Richardson compactification   Then Gazik [5]   has proved

the following result

__Theorem__ 3 1   R(S) is regular if and only if $v = Cl(v)$, for each

non-convergent ultrafilter v on S

Here $Cl(v)$ denotes the filter whose base is $Cl(V)$ for V in v,

where closure has been taken in S

If S is a principal convergence space, then, by construction,

R(S) is also principal   Therefore, if R(S) is regular, then by

Theorem 1 23, Theorem 1 17 and Theorem 1 24, R(S) is a compact

Hausdorff topological space, and hence, S is a Tychonoff topological

space   If S is a Tychonoff topological space, and $v = Cl(v)$ for

every non-convergent ultrafilter v on S, then as argued above,

R(S) is a topological compactification of S   And hence, by virtue

of its property in regard to the extension of continuous functions,

we see that $R(S) = \beta(S)$, where $\beta(S)$ is the Stone-Čech compacti-

fication of S   On the other hand, if S is a Tychonoff topological

space, and $R(S) = \beta(S)$, then R(S) is regular, and hence, $v = Cl(v)$

for every non-convergent ultrafilter v on S, by Theorem 3 1 above

The following result is due to Gazik [5]

<u>Theorem</u> 3 2   If S is a Tychonoff topological space, then

$R(S) = \beta(S)$ if and only if $v = Cl(v)$ for every non-convergent

ultrafilter v on S

2   *As The Largest Hausdorff Compactification*

In this section we will obtain a necessary and sufficient

condition that the Richardson compactification be the largest

Hausdorff compactification of the underlying convergence space

<u>Characterization</u>

<u>Theorem</u> 3 3   If S is a Hausdorff convergence space, then the

Richardson compactification (X_1, i) of S is the largest Hausdorff

compactification of S if and only if the following two conditions

are satisfied

(a)   S is locally compact,

(b) $\overset{\bullet}{v} = v \wedge i(v)$, for each non-convergent ultrafilter v on S

<u>Proof.</u>   Let (X_1, i) be the largest Hausdorff compactification

of S   Then, by Theorem 2 7, S is locally compact   Let us define

a new convergence structure on X_1 (and denote the resulting new

space by X_2) as follows

let ϕ be a filter on X_1, then

ϕ X_2-convergences to x if and only if ϕ

X_1 - converges to x,

ϕ X_2-converges to v if and only if $\phi \geq v \wedge \psi$, where ψ is an

ultrafilter on X_1 which X_1-converges to v

It is easy to see that $(X_2, 1)$ is a Hausdorff compactification

of S and $(X_2, 1) \geq (X_1, 1)$ But by assumption $(X_1, 1) \geq (X_2, 1)$,

hence by Theorem 1 21, $(X_1, 1) = (X_2, 1)$ and therefore, $X_1 = X_2$

This implies that a filter ϕ on X_1 will X_1-converge to v in

$X_1 - 1(S)$ if and only if $\phi \geq v \wedge \psi$, where ψ is an ultrafilter

which X_1-converges to v Again, we know $1(v) \geq \hat{v}$, and hence

$1(v) \geq v \wedge \psi$ But $1(v)$ is an ultrafilter on X_1 and $1(v) \neq v$,

hence $i(v) = \psi$ Since $\hat{v}$ X_1-converges to v, we conclude that

$\hat{v} \geq v \wedge i(v)$ But on the other hand $v \geq \hat{v}$ and therefore,

$v \wedge 1(v) \geq \hat{v}$ Thus we have shown that $\hat{v} = v \wedge 1(v)$

Next, to prove that the two conditions are also sufficient,

we will show that under these conditions every continuous function

k from S into a compact Hausdorff convergence space Y has a

unique continuous extension f from X_1 onto the closure of

$k(S)$ in Y, such that $f \circ i = k$

$$X_1 \xrightarrow{\quad f \quad} Y$$

with arrows $\imath$ from S to X_1 and k from S to Y.
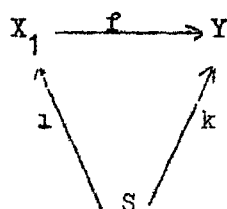
Let us define a function f from X_1 to Y as follows

$f(x) = k(x)$, for x in S

$f(v) = \lim k(v)$, for non-convergent ultrafilter v on S

Now, it is easy to see that f is a well defined function onto the closure of $k(S)$ in Y such that $f \circ \imath = k$ We will show that f is continuous Let ϕ be a filter on X_1 which X_1- converges to x Then, condition (a) implies that $\imath(S)$ belongs to ϕ, and therefore, $\imath^{-1}(\phi \cap \imath(S))$ S-converges to x Hence, $k \circ \imath^{-1}(\phi \cap \imath(S))$ Y-converges to $k(x)$ That is $f(\phi)$ Y-converges to $f(x)$ Let $\phi \ X_1$-converge to v Then $\phi \geq \hat{v}$, and condition (b) implies that $\phi \geq v \wedge \imath(v)$ Therefore, $f(\phi) \geq f(v \wedge \imath(v)) = f(v) \wedge f \circ \imath(v) = f(v) \wedge k(v)$, which implies $f(\phi)$ Y-converges to $f(v)$ Therefore, f is continuous, and since, uniqueness follows from Theorem 1 20, this completes the proof of the theorem

Remark

If a Hausdorff convergence space S has the largest Hausdorff compactification $L(S)$, then by Theorem 2 8, S has

only finitely many non-convergent ultrafilters   Then the

Hausdorff compactification (Y,k) of S constructed in the

proof of the Theorem 2 8, is the largest Hausdorff compactification

of S, and hence, $L(S) = (Y,k)$ Now, since (Y,k) has the

property that every continuous function from S into a compact

Hausdorff space has a unique extension to Y, so does $L(S)$ On

the other hand, if some Hausdorff compactification of S has

this property of extension of continuous functions, then it is

obviousely the largest Hausdorff compactification of S   Hence,

the largest Hausdorff compactification is characterized by this

property   Therefore, whenever the Richardson compactification

is the largest Hausdorff compactification of S, this compacti-

fication is characterized by the property of extension of continuous

functions

CHAPTER IV

ON REGULAR COMPACTIFICATIONS

Introduction

In this chapter we study regular compactifications of convergence spaces   In the last chapter we have seen a condition under which the Richardson compactification of convergence space becomes regular   But the more pertinent question is to characterize the class of convergence spaces which possess regular compactifications   Such a characterization has been obtained by Richardson and Kent [15]   Their condition is in terms of pretopological modification of the convergence space   They have shown that if a convergence space has a regular compactification, then its pretopological modification is a Tychonoff topological space, and corresponding to each Hausdorff topological compactification of this Tychonoff space, we can construct one regular compactification of the convergence space   They have further shown that the regular compactification corresponding to Stone-Cech compactification of the Tychonoff space is in fact the largest regular compactification Hence, each convergence space having regular compactifications, also possesses the largest regular compactification   In section one of this chapter we mention the results of Richardson and Kent

without proof   In section two we investigate the problem of the

smallest regular compactification   We prove that a convergence

space possessing regular compactifications has the smallest

regular compactification if and only if its pretopological

modification is a locally compact topological space   We also

define an equivalence relation on the class of all regular

compactifications of a convergence space, and show that each

equivalence class has a largest member and a smallest member

Each of this equivalence class, in fact, corresponds to a Hausdorff

topological compactification of the pretopological modification of

the convergence space, and the largest member of this equivalence

class is the one obtained by Richardson and Kent

1   *Largest Regular Compactification*

Richardson and Kent [15]   have shown that the closure

operator of a compact regular convergence space S is idempotent

(Theorem 1 23), and hence, by Theorem 1 17, the pretopological

modification of S is, in fact, the topological modification of

S   Further they have shown that in this case the pretopological

modification of S is also Hausdorff (Theorem 1 24)   This leads

to the following result

Theorem 4 1   If (T,f) is a regular compactification of S,

then $(\pi T, f)$ is a Hausdorff topological compactification of πS

The main results of Richardson and Kent are the following

Theorem 4 2   A regular convergence space has regular compactification if and only if πS is a completely regular topological space, and each ultrafilter which is finer than the neighbourhood filter at x S-converges to x for all x in S

Theorem 4 3   If a convergence space has a regular compactification, then it has a Stone-Čech regular compactification

2   *Smallest Regular Compactification*

In this section we will obtain a characterization of the class of convergence spaces which possess smallest regular compactifications

Relation between the compactifications of S and πS

We call a regular convergence space having regular compactifications as R-convergence space   In this section S will always denote an R-convergence space   If (P,g) is a regular compactification of S, then as mentioned in Theorem 4 1, $(\pi P, g)$ is a Hausdorff topological compactification of πS Let K denote the set of all regular convergence structures γ on P which satisfy the following two conditions

(a) γ coincides with πP relative to ultrafilter

convergence,

(b) if $g(S)$ belongs to a filter ϕ on P, then ϕ γ-converges

to y in $g(S)$ if and only if $g^{-1}(\phi \cap g(S))$ S-converges

to $g^{-1}(y)$

These two conditions are consistent, because, S being

an R-convergence space, every ultrafilter finer than the neighbour-

hood filter at x S-converges to x for all x in S   Let us

denote by P_0 and P_1 the convergence spaces consisting of the

set P equipped with the inf and sup of convergence structures

in K respectively   Now, it is easy to see that (P_0, g) and

(P_1, g) are regular compactifications of S   Further, $\pi P_0 =$

$\pi P_1 = \pi P$, and $P_1 \geq P \geq P_0$ On the other hand, if P' is a

regular convergence space such that $P_1 \geq P' \geq P_0$, then $\pi P_1 =$

$\pi P' = \pi P_0$, and (P', g) is a regular compactification of S

For the rest of the section (P_0, g) and (P_1, g) will stand for the

regular compactifications of S as obtained here for a given

regular compactification (P, g) of S

On (T_1, f)

If (T, f) is a Hausdorff topological compactification of

the Tychonoff space πS, then Richardson and Kent [15] have

constructed a regular compactification (T_1,f) of S By

construction $T = \pi T_1$ and T_1 is the finest regular convergence

space on the set T satisfying the two conditions (a) and (b)

above for γ and πP replaced by T_1 and T respectively

Now, we prove the following result

Theorem 4 4   If (P,g) is a regular compactification of S

and $(T,f) \geq (\pi P,g)$ as Hausdorff topological compactifications

of πS, then $(T_1,f) \geq (P,g)$ as regular compactifications of S

Proof   Let h be the continuous function from T onto πP
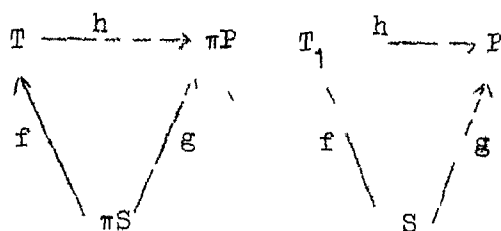
such that $h \circ f = g$

We will show that h is a continuous function from T_1 onto P

If not, there exists a filter ϕ_1 on T such that ϕ_1

T_1-converges to some point x_1 in T, but $h(\phi_1)$ does not

P-converge to $h(x_1)$ We define a new convergence structure

on the set T (and denote the resulting space by T_1') as follows

let ϕ be a filter on T, then

$\phi T_1'$-converges to $y \neq x_1$ in T if and only if

ϕT_1-converges to y,

$\phi T_1'$-converge to x_1 if and only if ϕ T_1-converges

to x_1 and $h(\phi)$ P-converges to $h(x_1)$

Now, $T_1' > T_1$ implies that T_1' is Hausdorff To see

that T_1' coincides with T_1 (and hence with T) relative to

ultrafilter convergence, we proceed as follows   Since $T_1' > T_1$,

we need only prove that if an ultrafilter ϕ on T is

T_1-convergent to y, then ϕ is T_1'-convergent to y If

$y \neq x_1$, by definition of T_1', this is true   It $y = x_1$, then

ϕT_1-converges to x_1 implies ϕ T-converges to x_1, because

T_1 and T coincide relative to ultrafilter convergence   Since h

is a continuous function from T to πP, $h(\phi)$ πP-converges

to $h(x_1)$ Now $P \geq \pi P$, and both P and πP are compact,

Hausdorff spaces, and therefore P and πP coincide relative

to ultrafilter convergence   Since $h(\phi)$ is an ultrafilter on P,

we see that $h(\phi)$ P-converges to $h(x_1)$ Therefore, by definition

$\phi T_1'$ - converges to x_1 This condition further implies that the

closure operators of T_1', T_1 and T are same   We show that

T_1' is regular as follows (1) if a filter ϕ on T is

T_1'-convergent to $y \neq x_1$, then ϕ is T_1-convergent to y,

since T_1 is regular, $\mathrm{Cl}_{T_1} \phi$ T_1-converges to y But T_1-closure

is same as T_1'-closure, hence $Cl_{T_1'}\phi$ T_1'-converges to y by

definition, (11) if ϕ T_1'-converges to x_1, then ϕ T_1-converges

to x_1, and since T_1 is regular, $Cl_{T_1}\phi$ T_1-converges to x_1,

hence $Cl_{T_1'}\phi$ T_1-converges to x_1 Also $h(\phi)$ P-converges to

$h(x_1)$, implies $Cl_P h(\phi)$ P-converges to $h(x_1)$ As pointed out

above P and πP coincide relative to ultrafilter convergence,

and hence, have same closure operators Therefore, $Cl_{\pi P} h(\phi)$

P-converges to $h(x_1)$ Since h is continous from T to πP,

Theorem 1 14 gives that $h(Cl_T \phi) \geq Cl_{\pi P} h(\phi)$ Therefore,

$h(Cl_T \phi)$ P-converges to $h(x_1)$ That is, $h(Cl_{T_1'}\phi)$ P-converges

to $h(x_1)$ Hence, by definition $Cl_{T_1'}\phi$ T_1'-converges to x_1

That is, T_1' is regular Next, let $f(S)$ belongs to a filter ϕ

on T, then ϕ T_1'-converges to some y in $f(S)$ if and only if

$f^{-1}(\phi \cap f(S))$ S-converges to $f^{-1}(y)$ To see this, we note that

if $f(S)$ belongs to ϕ and ϕ T_1'-converges to y in $r(S)$,

then ϕ T_1-converges to y in $f(S)$, and by virtue of property

(a) of T_1, we know that $f^{-1}(\phi \cap f(S))$ S-converges to $f^{-1}(y)$

On the other hand if $f^{-1}(\phi \cap f(S))$ S-converges $f^{-1}(y)$, then

again by the same property, ϕ T_1-converges to y And also

$g \circ f^{-1}(\phi \cap f(S))$ P-converges to $g \circ f^{-1}(y)$, because (P,g) is a

compactification of S That is $h(\phi)$ P-converges to $h(y)$

Hence, ϕ T_1'-converges to y Therefore, we see that T_1' is

a regular convergence space on the set T satisfying the two

conditions (a) and (b) for γ and πP replaced by T_1' and T

respectively   Also $T'_1 > T$, but this is a contradiction of

the fact that T_1 is the finest such convergence structure on

the set T Hence, h is a continuous function from T_1 onto

P such that $h \circ f = g$ That is, $(T_1, f) \geq (P, g)$ as regular

compactifications of S This completes the proof of the theorem

The result of Richardson and Kent

If (T, f) is the Stone-Čech compactification of the

Tychonoff space πS, and if (P, g) is a regular compactification

of S, then $(T, f) \geq (\pi P, g)$ as topological compactifications

of πS Hence, by Theorem 4 4 $(T_1, f) \geq (P, g)$ as regul r compacti-

fications of S This implies that (T_1, f) is the largest

regular compactification of S Hence, each R-convergence space

has a Stone-Čech regular compactification

On (T_0, f)

Let us denote by T_0 the convergence space consisting

of the set T equipped with the inf of all regular convergence

structures γ on T satisfying the two conditions (a) and (b)

above for πP replaced by T Then as pointed out above (T_0, f)

is a regular compactification of S such that $\pi T_1 = \pi T_0 = T$

Below we give an alternative description of T_0

__Theorem__ 4 5 T_0 is the finest convergence space on the set T

having the following properties

(i)   if y does not belong to $f(S)$, then $V_T(y)$ T_0-converges

   to y,

(ii)   if y belongs to $f(S)$ and trace of $V_T(y)$ on $T-f(S)$

   exists, then $Cl_T \phi \wedge [V_T(y) | (T-f(S))]$ T_0-converges to y,

   where $f(S)$ belongs to the filter ϕ on T and

   $f^{-1}(\phi \cap f(S))$ S-converges to $f^{-1}(y)$

(iii)   if y belongs to $f(S)$ and trace of $V_T(y)$ on $T-f(S)$

   does not exist, then $Cl_T \phi$ T_0-converges to y, where

   $f(S)$ belongs to the filter ϕ on T and $f^{-1}(\phi \cap f(S))$

   S-converges to $f^{-1}(y)$

__Proof__   The proof follows in two steps, firstly we will show that

T_0 satisfies the conditions (i), (ii) and (iii) of the theorem,

and secondly that T_0 is the finest such convergence space   We

see that condition (iii) is satisfied by T_0 by construction

Suppose T_0 does not satisfy the condition (i), that is, there

exists a point y_0 in $T-f(S)$ such that $V_T(y_0)$ does not

T_0-converge to y_0 Then let us define a new convergence structure

on $T($ and denote it by T_0') as follows

let ϕ be a filter on T, then

$\phi T_0'$-converges to $x \neq y_0$ if and only if ϕ

T_0-converges to x,

$\phi T_0'$-converges to y_0 if and only if $\phi \geq V_T(y_0)$

Then obviously $T_o > T_o'$. We see that if ϕ is an ultrafilter T_o'-converging to y_o, then $\phi \geq V_T(y_o)$, and therefore, ϕ T-converges to y_o. Hence, we can verify that T_o' coincide with T_o (and therefore with T) relative to ultrafilter convergence. This implies that T_o' is Hausdorff and closure operators of T_o', T_o and T are the same. Now, let a filter ϕ T_o'-converges to y_o implies $\geq V_T(y_o)$, and therefore ϕ T-converges to y_o, and therefore, $\mathrm{Cl}_T \phi$ T-converges to y_o, that implies $\mathrm{Cl}_T \phi \geq V_T(y_o)$, therefore $\mathrm{Cl}_{T_o'} \phi \geq V_T(y_o)$, and hence, $\mathrm{Cl}_{T_o'} \phi$ T_o'-converges to y_o. This shows that T_o' is regular. Since y_o does not belong to $f(S)$, we see that T_o' satisfies the conditions (a) and (b) above for γ and πP replaced by T_o' and T respectively. Since $T_o > T_o'$, and T_o is the coarsest convergence space satisfying these conditions, we get a contradiction. Hence, T_o satisfies condition (i) of the theorem. Next, suppose T_o does not satisfy the condition (ii). Then there exists a point y_o in $f(S)$ and a filter ϕ_o on T such that $f(S)$ belongs to ϕ_o and $f^{-1}(\phi_o | f(S))$ S-converges to $f^{-1}(y_o)$, and $V_T(y_o) \cap (T-f(S))$ exists, but $\mathrm{Cl}_T \phi_o \wedge [V_T(y_o) \cap (T-f(S))]$ does not T_o-converge to y_o. Let us define a new convergence structure on the set T (and denote it by T_o') as follows

let ϕ be a filter on T, then

$\phi T_o'$-converges to $y \neq y_o$ if and only if ϕ

T_o- converges to y,

ϕ T_o'-converges to y_o if and only if either

ϕ T_o-converges to y_o or $\phi \geq Cl_T \phi_o \wedge [V_T(y_o) \cap (T-f(S))]$

It is simple to see that T_o' is a convergence space, for this we need only note that ϕ_o T-converges to y_o and therefore $y_o \geq Cl_T \phi_o$. By construction $T_o > T_o'$. To prove that T_o'-coincides with T_o (and therefore with T) relative to ultrafilter convergence, we need only note that $[V_T(y_o) \cap (T-f(S))] \geq V_T(y_o)$, and hence $[V_T(y_o) \cap (T-f(S))]$ T-converges to y_o. This implies that T_o' is Hausdorff and closure operators of T_o', T_o and T are same. This in turn implies that T_o' is regular, because $Cl_T \phi_o \wedge [V_T(y_o) \cap (T-f(S))]$ is a T-closed filter on T. To see that T_o' satisfies condition (b) above for γ and πP replaced by T_o' and T respectively, we need only show that if $f(S)$ belongs to ϕ and $\phi \geq Cl_T \phi_o \wedge [V_T(y_o) \cap (T-f(S))]$, then $f^{-1}(\phi \setminus f(S))$ S-converges to $f^{-1}(y_o)$. Let ϕ' be an ultrafilter on T such that $\phi' \geq \phi$. Since $f(S)$ belongs to ϕ', we see that $\phi' \not\geq [V_T(y_o) \cap (T-f(S))]$. Hence $\phi' \geq Cl_T \phi_o$. This implies $\phi \geq Cl_T \phi_o$. Now, $f^{-1}(\phi_o \setminus f(S))$ S-converges to $f^{-1}(y_o)$ implies ϕ_o T_o-converges to y_o, and T_o is regular, therefore $Cl_T \phi_o$ T_o-converges to y_o. Hence, ϕ T_o'-converges to y_o. Thus T_o' satisfies the conditions (a) and (b) above for γ and πP replaced by T_o' and T respectively and also $T_o > T_o'$. This gives a contradiction because T_o is the coarsest such space. Therefore, T_o satisfies

condition (ii) of the theorem   Next, we will show that if some
convergence space T_0' on T satisfies conditions (i), (ii) and
(iii) of the theorem, then $T_0 \geq T_0'$ Let a filter ϕ on T is
T_0-convergent to y not in $f(S)$ Then T being pretopological
modification of T_0, ϕ T-converges to y, and therefore $\phi \geq V_T(y)$
Hence, by condition (i) ϕ T_0'-converges to y Let ϕ T_0-converges
to y in $f(S)$, and let (A_1) $f(S)$ belongs to ϕ and
$f^{-1}(\phi \| f(S))$ S-converges to $f^{-1}(y)$, then ϕ T_0'-converges to y by
condition (ii) or (iii) according as $V_T(y) \| (T-f(S))$ exists
or not; (A_2) if $T-f(S)$ belongs to ϕ, then $V_T(y) \cap (T-f(S))$
exists and $\phi \geq [V_T(y) \| (T-f(S))]$ Hence, by condition (ii)
ϕ T_0'-converges to y, (A_3) if neither $f(S)$ nor $T-f(S)$ belongs
to ϕ, then $V_T(y) \cap (T-f(S))$ does exist   Let $\phi_1 = [\phi \cap f(S)]$
and $\phi_2 = \lceil \phi \| (T-f(S)) \rceil$ Then, $\phi = \phi_1 \wedge \phi_2$ Now, $f(S)$
belongs to $\phi_1 \geq \phi$, therefore ϕ_1 T_0-converges to y and hence
$f^{-1}(\phi_1 \cap f(S))$ S-converges to $f^{-1}(y)$ And $T-f(S)$ belongs to
$\phi_2 \geq \phi \geq V_T(y)$, therefore, $\phi_2 \geq [V_T(y) \cap (T-f(S))]$ Hence, by
condition (ii) ϕ T_0'-converges to y Thus we conclude that T_0
is the finest convergence space on T satisfying (i), (ii) and (iii)
This completes the proof of the theorem

Now using Theorem 4 5, we will prove a theorem for (T_0, f) which
is analogous to Theorem 4 4 for (T_1, f)
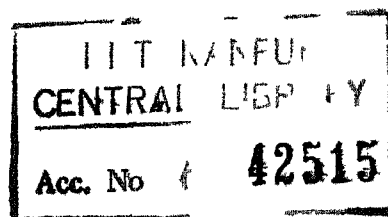
Theorem 4 6   If (P,g) is a regular compactification of S and
$(\pi P,g) \geq (T,f)$ as Hausdorff topological compactifications of πS,
then $(P,g) \geq (T_0,f)$ as regular compactifications of S

Proof   Let h be the continuous function from πP onto T
such that $h \circ g = f$ We will show that h is continuous from
P onto T_0
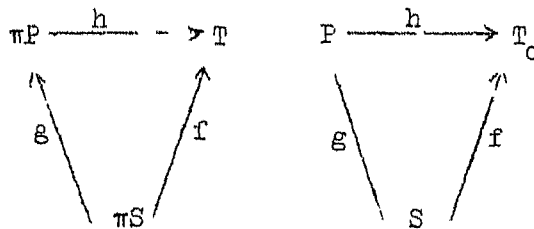
Let ϕ be a filter on P which P-converges to some y not in
$g(S)$ Then ϕ πP-converges to y, and hence, $h(\phi)$ T-converges
to $h(y)$ and this implies that $h(\phi) \geq V_T(h(y))$ Now by
Theorem 1 22, $h(y)$ does not belong to $f(S)$, and therefore, by
condition (i) of Theorem 4 5, $h(\phi)$ T_0-converges to $h(y)$ Now,
let ϕ P-converge to y in $g(S)$ Then, (A_1) if $g(S)$ belongs
to ϕ, then this implies $g^{-1}(\phi \mid g(S))$ S-converges to $g^{-1}(y)$
This implies, $f \circ g^{-1}(\phi \cap g(S)) = h(\phi)$ T_0-converges to $f \circ g^{-1}(y)$
$= h(y)$, (A_2) if $g(S)$ does not belong to ϕ, let $\phi_1 = [\phi \cap (P-g(S))]$
and if the trace of ϕ on $g(S)$ also exists, then $\phi_2 = [\phi \cap g(S)]$
Then, $\phi = \phi_1 \wedge \phi_2$ Since ϕ P-converges to y, $h(\phi) \geq V_T(h(y))$
By Theorem 1 22, since $g(S)$ does not belong to ϕ, $f(S)$ does not

belong to $h(\phi)$ Hence $V_T(h(y)) \cap (T-f(S))$ exists   Now,

$\phi_1 \geq \phi$ which P-converges to y implies $h(\phi_1)$ T-converges to

$h(y)$ This implies, $h(\phi_1) \geq V_T(h(y))$ Since P-g(S) belongs

to ϕ_1, T-f(S) belongs to $h(\phi_1)$ by Theorem 1 22 again   Hence,

$h(\phi_1) \geq [V_T(h(y)) \cap (T-f(S))]$ Also, $g(S)$ belongs to $\phi_2 \geq \phi$

Hence ϕ_2 P-converges to y, this implies $g^{-1}(\phi_2 \cap g(S))$ S-converges

to $g^{-1}(y)$ This implies $f \circ g^{-1}(\phi_2 \cap g(S))$ T_0-converges to

$f \circ g^{-1}(y)$ Hence, $h(\phi_2)$ T_0-converges to $h(y)$ Again, by

Theorem 1 22, $f(S)$ belongs to $h(\phi_2)$ Hence, we see that

$h(\phi) \geq h(\phi_1) \wedge h(\phi_2) \geq [V_T(h(y)) \cap (T-f(S))] \wedge Cl_T(h(\phi_2))$ Thus,

by condition (ii) of Theorem 4 5, $h(\phi)$ T_0-converges to $h(y)$

Hence, h is continuous from P onto T_0 and $h \circ g = f$ Therefore,

$(P,g) \geq (T_0,f)$ This completes the proof

Characterization

If πS is a locally compact Hausdorff topological space

and (T,f) is the Alexandroff one point Hausdorff topological

compactification of πS, then from Theorem 4 6, we find that,

if (P,g) is a regular compactification of S, then $(\pi P,g) \geq (T,f)$,

and therefore, $(P,g) \geq (T_0,f)$ That is, (T_0,f) is the smallest

regular compactification of S   On the other hand, if S has the

smallest regular compactification say (P,g), then we will show

that $(\pi P,g)$ is the smallest Hausdorff topological compactification

of the Tychonoff space πS, and hence, πS is a locally compact

Hausdorff topological space   Let (T,f) be an arbitrary

Hausdorff topological compactification of πS Let (T_1,f)

be the Richardson-Kent regular compactification of S, corresponding

to (T,f) of πS Then, $(T_1,f) \geq (P,g)$ Let h be the

continuous function from T_1 onto P such that $h \circ f = g$ Then,

from Theorem 1 16, h is continuous from πT_1 onto πP But

$\pi T_1 = T$, hence $(T,f) \geq (P,g)$

This leads us to our main result

Theorem 4 7   If S is an R-convergence space, then S has the

smallest regular compactification if and only if πS is a locally

compact Hausdorff topological space

Equivalence classes

We will define an equivalence relation on the set of all

regular compactifications of an R-convergence space , as follows

if (P,g) and (Q,f) are two regular compactifications of S,

then $(P,g) \sim (Q,f)$ if and only if $(\pi P,g) = (\pi Q,f)$ as

Hausdorff topological compactifications of the Tychonoff space πS

This gives us the following result

Theorem 4.8   If S is an R-convergence space, then each

equivalence class of regular compactifications of S, has a

largest member and a smallest member

<u>Proof</u>   Let $(P,g) \sim (Q,f)$ Since $P_1 \geq P \geq P_0$ and $Q_1 \geq Q \geq Q_0$, we need only show that $(P_1,g) = (Q_1,f)$ and $(P_0,g) = (Q_0,f)$ First of these follows from Theorem 4 4   and the second from Theorem 4 6

ON m-ULTRACOMPACTIFICATIONS

Introduction

In this chapter our aim is to introduce m-ultracompactification
This is a generalisation of the Richardson compactification   We use
Richardson's technique to construct the m-ultracompactification
m-ultracompactness for T_1 topological spaces have been defined by
J van der Slot [16] , for an infinite cardinal m $\aleph_0$-ultracompactness
is equivalent to compactness   Realcompactness and $\aleph_1$-ultracompact-
ness for countably compact normal spaces are equivalent   This definition
is suitable for extension to convergence spaces and works well with
Richardson method of compactification   In the first section we find
condition for a convergence space to have smallest Hausdorff m-ultra-
compactification   In section two we construct a Hausdorff m-ultracompacti-
fication in a manner analogous to Richardson compactification   We
show that if S is the convergence space and ρS the Hausdorff
m-ultracompactification and f is a continuous function from S into
a Hausdorff regular m-compact space, then f has a unique continuous
extension to ρS Then we find conditions under which ρS becomes
the largest Hausdorff (regular) m-ultracompactification of S When
S is a Tychonoff topological space, we do not know condition under

which ρS becomes topological   It is mentioned here as an open

problem   Two more open problems are also mentioned

1   *Smallest Hausdorff m-ultracompactification*

In this section we characterize the class of Hausdorff

convergence spaces having smallest *m*-ultracompactification   We

first note that compact spaces are *m*-ultracompact for each

infinite cardinal *m*   Hence each Hausdorff convergence space

has a one point Hausdorff *m*-ultracompactification (the one point

Hausdorff compactification constructed in Chapter One will serve

the purpose)

Local *m*-ultracompactness

Definition 5 1   A Hausdorff convergence space S is locally

m-ultracompact if and only if it is open in each of its Hausdorff

m-ultracompactifications

Characterization

Theorem 5 2   A Hausdorff convergence space has smallest Hausdorff

m-ultracompactification if and only if S is locally *m*-ultracompact

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<u>Proof</u>   Let a Hausdorff convergence space S have the smallest

Hausdorff m-ultracompactification (T',J) and let $(T, 1)$ be

the one point Hausdorff compactification of S   Then, $(T,1) \geq (T',J)$

By Theorem 1 22, we conclude that (T',J) is a one point m-ultra-

compactification and hence S is open in T'   Another application

of Theorem 1 22 together with Theorem 1 15, enables us to conclude

that S is open in each of its m-ultracompactifications

　　　Next, assuming S to be locally m-ultracompact, we can

show that the one point compactification $(T, 1)$ is the smallest

m-ultracompactification of S   The proof follows exactly in the

same line as in the proof of Theorem 2 2

2 m-ultracompactification

<u>Construction</u>

　　　Let S be a Hausdorff convergence space and ρS the set

of all x for x in S and all non-convergent m-ultrafilters

on S   If $\overline{F}$ is a filter on S, then we define a filter $\hat{F}$

on ρS as the one having basis $\{\hat{F} | F \in F\}$, where $\hat{F} = \{H \in \rho S | F \in H\}$

We define a convergence structure on ρS as follows

let ϕ be a filter on S , then

ϕ ρS-converges to x for x in S if and only if $\phi \geq \hat{F}$

　　for some filter F on S which S-converges to x

ϕ ρS-converges to v, for non-convergent

m-ultrafilter v on S if and only if $\phi \geq \hat{v}$ We note that

$\hat{F}_1 \mid\mid \hat{F}_2 = (F_1 \cap F_2)^{\wedge}$, $\hat{F}_1 \cup \hat{F}_2 = (F_1 \cup F_2)^{\wedge}$, $x \geq \lambda$, $v \geq \hat{v}$, using

these we can see that ρS is a Hausdorff convergence space To

show that ρS is m-ultracompact, let ϕ be an ultrafilter on

ρS and $C_{\rho S}$ denote the set of all ρS-closed subsets and $\phi \mid\mid C_{\rho S}$

has m-intersection property Let $F = \{F \subset S \mid \hat{F} \in \phi\}$, then F

is ultrafilter on S e will show that F is an m-ultrafilter

on S Let $F_1 \in F \cap C_S$, where $\iota \in I$ and $|I| \leq m$ We will

first show that if A is S-closed then $\hat{A}$ is ρS-closed Let ψ

be a filter on ρS containing $\hat{A}$ which ρS-converges to some H

in ρS (1) If $H = x$, $\psi \geq \hat{F}$ for some F S-converging to x

This implies $\hat{A} \cap \hat{F} \neq \emptyset$ for every F in F This implies

$A \cap F \neq \emptyset$ for every F in F Let G be a filter on S containing

A and $G \geq F$, then G S-converges to λ and hence x belongs

to $Cl_S A = A$ Therefore, $H = x \in \hat{A}$ (11) If $H = v$ for some v

in $\rho S - S$, then $\psi \geq \hat{v}$ This implies $\hat{A} \cap \hat{V} \neq \emptyset$ for every V

in v Hence $A \mid\mid V \neq \emptyset$ for every V in v, but v is an

ultrafilter on S, therefore A belongs to v Hence $H = v \in \hat{A}$

Now since F_1's are S-closed $\hat{F}_1$'s are ρS-closed And

$\hat{F}_1 \in \phi$ by definition of F Hence $\hat{F}_1 \in \phi \cap C_{\rho S}$ Now

$\phi \cap C_{\rho S}$ has m-intersection property, therefore $\cap \hat{F}_1 \neq \emptyset$

Let $H \in \mid\mid \hat{F}_1$, then $H \in \hat{F}_1$ for each i, that is, $F_1 \in H$ for

each ι Now each $H \in \rho S$ is an m-ultrafilter on S, hence

$\cap F_1 \neq \emptyset$ Hence F has m-intersection property By construction

$\phi \geq \hat{F}$ Now F on S either S-converges to some x in S or F is a non-convergent m-ultrafilter on S, in the first case ϕ ρS-converges to x and in the second case ϕ ρS-converges to $F \in \rho S-S$ Let us define a function i from S to ρS as $i(x) = x$ We note that $i(F) = \hat{F} \cap i(S)$ for filter F on S From this we can easily see that i is an isomorphism from S to ρS Further if $v \in \rho S-S$, then $i(v) \geq \hat{v}$ ρS-converges to v and contains $i(S)$, hence $i(S)$ is dense in ρS Finally let f be a continuous function from S into a regular m-ultracompact convergence space T', then let us define f_1 from ρS to T' as follows

$f_1(x) = f(x)$, for x in S

$f_1(v) = \lim f(v)$, for v in $\rho S-S$

We note that if $v \in \rho S-S$, then $v \cap C_S$ has m-intersection property and v is a non-convergent ultrafilter on S We will show that $f(v)$ is an m-ultrafilter on T', then $f(v)$ will T'-converge to a unique point, that point is defined as $f_1(v)$ Let $A_i \in f(v) \cap C_{T'}$, $i \in I$ and $|I| \leq m$ Then there exist F_i's in v such that $f(F_i) \frown A_i$ Therefore $f^{-1}f(F_i) \supset f^{-1}(A_i)$, that is, $F_i \subset f^{-1}f(F_i) \supset f^{-1}(A_i)$ Now A_i's being T'-closed and f continuous from S to T', $f^{-1}(A_i)$ are S-closed and also belongs to v This implies, $\cap f^{-1}(A_i) \neq \emptyset$, therefore $\cap A_i \neq \emptyset$ Hence $f(v)$ is an m-ultrafilter on T' To see that f_1 is continuous from ρS to T', let ϕ be a filter on ρS which is

ρS-convergent to x , then $\phi \geq \hat{F}$ for some filter F on S which

S-converges to x Hence $f(F)$ T'-converges to $f(x)$ Since

T' is regular $Cl_{T'} f(F)$ T'-converges to $f(x)$ Now since

$f(x) = f_1(x)$, it is sufficient to show that $f_1(\phi) \geq Cl_{T'} f(F)$

In case ϕ ρS-converges to v in ρS-S, $\phi \geq \hat{v}$ as shown above

$f(v)$ is a m-ultrafilter on T', hence must converge, let it

T'-converges to y, then $Cl_{T'} f(v)$ also T'-converges to y

It is sufficient to show that $f_1(\phi) \geq Cl_{T'} f(v)$, because $f_1(v) =$

lim $f(v) = y$ Now, let ϕ be a filter on ρS such that $\phi \geq \hat{F}$

for some filter F on S, then we will show that $f_1(\phi) \geq Cl_{T'} f(F)$

Let F belongs to F , then there exists an F^* in ϕ such that

$F^* \supset \hat{F}$ We will show that $f_1(F^*) \sim Cl_{T'} f(F)$ Let v belongs

to F^*, then v belongs to $\hat{F}$, that is, F belongs to v, therefore

$f(F)$ belongs to $f(v)$ Since $f_1(v) = $ lim $f(v)$, $f_1(v)$ belongs

to $Cl_{T'} f(F)$ Hence f_1 is continuous from ρS to T'   Obviously

$f_1 \circ i = f$ Uniqueness of f_1 follows from Theorem 1 20

(ρS,i) as largest Hausdorff m-ultracompactification

__Theorem 5 4__   (ρS, i) is the largest Hausdorff m-ultracompacti-

fication of a Hausdorff convergence space S if and only if the

following two conditions are satisfied

(a)   S is open in ρS

(b) $\hat{v} = v \wedge i(v)$, for v in ρS-S

ρS-converges to v Therefore, $Cl_{\rho S}\, \iota(v)$ ρS-converges to v

This implies $Cl_{\rho S}\iota(v) \geq \hat{v}$ Hence if V belongs to v, then there

exists a V' in v such that $Cl_{\rho S}\iota(V') \sim \hat{v}$ We will show that

$Cl_S V' \subset V$ Let x belong to $Cl_S V'$, there exists a filter F

on S containing V' which S-converges to x This implies,

$\iota(F)$ ρS-converges to $\iota(r) = x$ and $\iota(V')$ belongs to $\iota(F)$

Hence x belongs to $Cl_{\rho S}\iota(V') \subset \hat{v}$ That says, x belongs to V

Therefore, we conclude that $Cl_S v \geq v$, hence $Cl_S v = v$ for each v

in ρS-S

Open Problems

(1)   Under what conditions will ρS become topological for

Tychonoff space S

(2)   Characterization of convergence spaces having largest

Hausdorff m-ultracompactifications

(3)   Characterization of convergence spaces having regular

m-ultracompactifications   When will these spaces have

largest (smallest) m-ultracompactifications

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PUBLICATIONS

The following two papers have appeared in print containing some of the results of the present thesis

(1) On smallest compactification for convergence spaces,

Proc Amer Math Soc 44 (1974), 225 - 230

The first section of this paper is on smallest Hausdorff compactification

(Section One of Chapter Two)

The second section of this paper is on smallest regular compactification

(Section Two of Chapter Four)

(2) On largest Hausdorff compactification for convergence spaces

Bull Austral Math Soc 12 (1975), 73 - 79

The first section of this paper is on Richardson compactification

(Section Two of Chapter Three)

The second section of this paper is on largest Hausdorff compactification

(Section Two of Chapter Two)

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MATH-1975-D-RAO-ON